Very Special Relativity is incompatible with Thomas precession

Suratna Das * and Subhendra Mohanty †

Physical Research Laboratory, Ahmedabad 380009, India.

Abstract

Glashow and Cohen make the interesting observation that certain proper subgroups of the Lorentz group like $HOM(2)$ or $SIM(2)$ can explain many results of special relativity like time dilation, relativistic velocity addition and a maximal isotropic speed of light. We show here that such $SIM(2)$ and $HOM(2)$ based VSR theories predict an incorrect value for the Thomas precession and are therefore ruled out by observations. In VSR theories the spin-orbital coupling in atoms turn out to be too large by a factor of 2. The Thomas-BMT equation derived from VSR predicts a precession of electrons and muons in storage rings which is too large by a factor of 10^3. VSR theories are therefore ruled out by observations.

* suratna@prl.res.in

† mohanty@prl.res.in

INTRODUCTION

Glashow and Cohen [1] have made the interesting observation that certain proper subgroups of the Lorentz group like $HOM(2)$ or $SIM(2)$ can explain many results of Special Relativity (SR) like time dilation, relativistic velocity addition and a maximal isotropic speed of light. Glashow and Cohen further suggest that particle physics models can be constructed with only $HOM(2)$ or $SIM(2)$ invariance (which they call VSR (Very Special Relativity)) which will give the same dynamics as the full Lorentz invariant theory. These subgroups ($HOM(2)$ or $SIM(2)$) of Lorentz group along with either P, CP or T generate the full Lorentz group [1]. An interesting application of this idea is to construct a $SIM(2)$ invariant mass term for the neutrino using only the standard model left handed neutrinos [2]. There are many theories that have been constructed based on the $HOM(2)$ or $SIM(2)$ algebra. In [3] it has been shown that Quantum Field Theories based on non-commutative space-times provide a setting for non-trivial realizations of the VSR algebras. VSR transformations can be generalized to curved space-time [4–6]. Super-symmetric theories based on SIM(2) algebra have also been constructed [7–9].

One interesting point to note is that VSR theories differ from other Lorentz violating theories where the Lorentz violation is governed by a small parameter and as this parameter approaches to zero, one regains the full Lorentz invariant theory. In VSR theories the Lorentz violation is obtained by replacing the Lorentz group by its proper subgroups.

Though the VSR theory has certain interesting consequences as mentioned above, it is worth noticing at this point that it will be incorrect to expect that a proper subgroup of the Lorentz group to reproduce all the results of the full Lorentz group. For an example consider the two $SU(2)$ subgroups generated by $N_i = \frac{1}{2}(J_i + iK_i)$ and $\tilde{N}_i = \frac{1}{2}(J_i - iK_i)$ of the Lorentz group. These two subgroups transform into each-other under parity. Any one of the subgroup augmented with parity can generate the full Lorentz group. However even for parity conserving processes it would be naive to expect all the results of the full Lorentz group $SO(3,1) \sim SU(2) \otimes SU(2)$ by taking only one of the $SU(2)$ subgroups generated by either N_i or $\tilde{N}_i$. The invariance of the Electromagnetic interaction involves the generators of both N_i and $\tilde{N}_i$ i.e. the full Lorentz group. Even for CP conserving processes one can not expect only one of these subgroups giving the same results as the full Lorentz group.

The above argument is also applicable for the subgroups ($HOM(2)$ and $SIM(2)$) of the

Lorentz group with which the VSR theories are constructed. So any result of which can be derived using the symmetries of the full Lorentz group need not necessarily follow by using the transformations of the VSR subgroups. It is a remarkable observation of Glashow and Cohen that many of the results of special relativity like time dilation, velocity addition and the constancy of the speed of light can be derived using only the $HOM(2)$ and $SIM(2)$ transformations and do not require the full $SO(3,1)$ group. We have shown in this paper that this does not hold for another classic result of Special Relativity namely Thomas precession.

We have arranged our paper in the following way : In Sec. () we verify the results of [1] that VSR theories can mimic the action of Lorentz transformations in boosts and in relativistic velocity addition, though the VSR transformation parameters required for velocity addition are not given either in [1] or in any follow up papers. In Sec. () we find however that one classic result of SR namely Thomas precession [10] which is tested in the spin-orbit interaction of atoms does not come out correctly in VSR. We show here that such $SIM(2)$ and $HOM(2)$ based VSR theories predict incorrect Thomas precession and are therefore ruled out by observations of the fine splitting of atomic spectra. A test of Thomas precession in a macroscopic setting is in the spin-precession in external magnetic field which is described by the Bargmann-Michel-Telegedi (BMT) equation [11]. We show in Sec. () that the VSR based theories lead to large corrections in the BMT equation and are ruled out by the observations of spin-precession in accelerators.

VERY SPECIAL RELATIVITY

The Cohen-Glashow Very Special Relativity (VSR) [1] is defined as symmetry under certain proper subgroups of Lorentz group. The minimal version of the VSR algebra contains, the subgroup $T(2)$ of the Lorentz group, which is generated by $T_1 = K_x - J_y$ and $T_2 = K_y + J_x$, where J_i and K_i $(i = x, y, z)$ are respectively generators of rotations and boosts. $T(2)$ is an Abelian subalgebra of Lorentz algebra $SO(1,3)$ and can be identified with the translation group on a two dimensional plane. The other larger versions of VSR are obtained by adding one or two Lorentz generators to $T(2)$, which have geometric realizations on the two dimensional plane. $E(2)$, the 3-parametric group of two dimensional Euclidean motion, generated by T_1, T_2 and J_z, with the structure $[T_1, T_2] = 0$, $[J_z, T_1] = T_2$, $[J_z, T_2] = -T_1$. $HOM(2)$, the group of orientation-preserving similarity transformations, or

homotheties, generated by T_1, T_2 and K_z, with the structure $[T_1, T_2] = 0$, $[T_1, K_z] = -T_1$, $[T_2, K_z] = -T_2$. $SIM(2)$, the group isomorphic to the four-parametric similitude group, generated by T_1, T_2, J_z and K_z. The explicit forms of the VSR generators are

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$J_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

The surprising result of VSR is that all the classic tests of Special Relativity like Michelson-Morley experiment, time dilation, constant isotropic maximal speed of light and velocity addition do not require the full Lorentz group but can be derived using the generators of just the $SIM(2)$ or $HOM(2)$ subgroups.

In VSR (Very Special Relativity) [1] one can transform a particle velocity in its rest frame $u_0 = (1, 0, 0, 0)$ to a moving frame with four velocity $u = (\gamma_u, \gamma_u u_x, \gamma_u u_y, \gamma_u u_z)$, where $\gamma_u = \frac{1}{\sqrt{1-\mathbf{u}^2}}$, by transformations of the $HOM(2)$ group

$$L(u)u_0 = u, \tag{2}$$

where

$$L(u) = e^{\alpha T_1} e^{\beta T_2} e^{\phi K_z} \tag{3}$$

and the parameters are given by [1]

$$\begin{aligned} \alpha &= \frac{u_x}{1-u_z}, \\ \beta &= \frac{u_y}{1-u_z}, \\ \phi &= -\ln[\gamma_u(1-u_z)]. \end{aligned} \tag{4}$$

We can also define the $HOM(2)$ transformation matrices in terms of the generators as

$$L(u) = e^{i\alpha T_1} e^{i\beta T_2} e^{i\phi K_z} \tag{5}$$

in that case one would have to include an extra factor of $-i$ with the generators given in Eq. (1). The parameters of the transformation remain the same as in Eq. (4) and are always real.

These parameters are chosen in [1] to give the same result as the Lorentz transformation in SR. The $HOM(2)$ transformation matrices are explicitly

$$e^{\alpha T_1} = \begin{pmatrix} 1+\frac{\alpha^2}{2} & \alpha & 0 & -\frac{\alpha^2}{2} \\ \alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \frac{\alpha^2}{2} & \alpha & 0 & 1-\frac{\alpha^2}{2} \end{pmatrix}, \tag{6}$$

$$e^{\alpha T_2} = \begin{pmatrix} 1+\frac{\beta^2}{2} & 0 & \beta & -\frac{\beta^2}{2} \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & -\beta \\ \frac{\beta^2}{2} & 0 & \beta & 1-\frac{\beta^2}{2} \end{pmatrix}, \tag{7}$$

$$e^{\phi K_z} = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix}, \tag{8}$$

which altogether yield the VSR transformation given in Eq. (3) as

$$L(u) = \begin{pmatrix} \gamma_u & \frac{u_x}{1-u_z} & \frac{u_y}{1-u_z} & \gamma_u \frac{u_z-u^2}{1-u_z} \\ \gamma_u u_x & 1 & 0 & -\gamma_u u_x \\ \gamma_u u_y & 0 & 1 & -\gamma_u u_y \\ \gamma_u u_z & \frac{u_x}{1-u_z} & \frac{u_y}{1-u_z} & \gamma_u \frac{1-u^2}{1-u_z} \end{pmatrix}. \tag{9}$$

It is to be noted here that by construction this transformation is quite different from the well known Lorentz boost given by

$$\Lambda(u) = \begin{pmatrix} \gamma_u & \gamma_u u_x & \gamma_u u_y & \gamma_u u_z \\ \gamma_u u_x & 1+\frac{(\gamma_u-1)u_x^2}{u^2} & \frac{(\gamma_u-1)u_x u_y}{u^2} & \frac{(\gamma_u-1)u_x u_z}{u^2} \\ \gamma_u u_y & \frac{(\gamma_u-1)u_x u_y}{u^2} & 1+\frac{(\gamma_u-1)u_y^2}{u^2} & \frac{(\gamma_u-1)u_y u_z}{u^2} \\ \gamma_u u_z & \frac{(\gamma_u-1)u_x u_z}{u^2} & \frac{(\gamma_u-1)u_y u_z}{u^2} & 1+\frac{(\gamma_u-1)u_z^2}{u^2} \end{pmatrix}. \tag{10}$$

The inverse transformation $L^{-1}(u) = e^{-\phi K_z} e^{-\beta T_2} e^{-\alpha T_1}$ takes a particle from a moving frame to its rest frame.

Velocity addition in VSR

Velocity addition law of Special Relativity (SR) is crucial in ensuring that the speed of light is maximal and same in all inertial reference frames. Hence to have an alternative theory of SR, the new theory should also produce the same result for velocity addition.

Suppose a particle is moving with velocity $\mathbf{u}$ in an inertial frame (S') which is moving with a velocity $\mathbf{v}$ with respect to another inertial frame (S). According to SR applying two successive boosts $\Lambda(v)\Lambda(u)$ on the rest frame of the particle gives the velocity addition law.

In VSR the velocity addition can not be given by successive $HOM(2)$ transformations $L(v)L(u)$ like that of in SR because the form of the VSR transformation operator depends upon the reference frame unlike Lorentz transformation of SR. So the operator which boosts a particle at rest with velocity $\mathbf{u}$ given in Eq. (3) is not the same as the operator $L(v,u)$ which transforms a particle with velocity $\mathbf{u}$ by a boost parameter $\mathbf{v}$. Such a transformation can be constructed in $HOM(2)$

$$L(v,u)u = w \tag{11}$$

(where w is the relativistic sum of u and v) with the general properties

$$L(v,0) = L(v) \tag{12}$$

and

$$L(0,u) = I. \tag{13}$$

For example the $HOM(2)$ transformation which boosts a particle with velocity $\mathbf{u} = (u_x, u_y, u_z)$ by the boost parameter $\mathbf{v} = (v_x, 0, 0)$ is

$$L(v,u) = e^{\alpha' T_1} e^{\beta' T_2} e^{\phi' K_z} \tag{14}$$

with parameters α', β', ϕ' chosen as,

$$\begin{aligned} \alpha' &= \frac{u_x - (u_x + v_x)\,\gamma_v}{u_z + (1 + u_x v_x)\,\gamma_v}, \\ \beta' &= 0, \\ \phi' &= -\frac{1 - u_z}{u_z + (1 + u_x v_x)\gamma_v}. \end{aligned} \tag{15}$$

One can check explicitly that

$$L(v,u)u = (\gamma_u \gamma_v (1 + u_x v_x),\ \gamma_u \gamma_v (u_x + v_x),\ \gamma_u u_y,\ \gamma_u u_z)^{\mathrm{T}}, \tag{16}$$

which is the correct relativistic velocity addition result.

To get the relativistic velocity addition when $\mathbf{v} = (0, v_y, 0)$ the parameters will be

$$\begin{aligned}
\alpha' &= 0 \\
\beta' &= \frac{u_y - (u_y + v_y)\,\gamma_v}{u_z + (1 + u_y v_y)\,\gamma_v}, \\
\phi' &= -\frac{1 - u_z}{u_z + (1 + u_y v_y)\gamma_v}.
\end{aligned} \tag{17}$$

However if $\mathbf{v} = (0, 0, v_z)$ i.e. S' is moving in the positive z direction with respect to S frame, then $L(v_z, u) = L(v_z)$ i.e. $\alpha' = 0$, $\beta' = 0$ and $\phi' = \phi$ and in this case $L(v_z)u$ will give the correct relativistic addition of velocities.

The transformation parameters α', β', ϕ' that boosts a particle with velocity $\mathbf{u} = (u_x, u_y, u_z)$ to $\mathbf{v} = (v_x, v_y, v_z)$ are algebraically complicated.

THOMAS PRECESSION AND SPIN-ORBITAL COUPLING IN VSR

Thomas precession is a result of the property of Lorentz transformation that two successive Lorentz boosts along different directions can be combined as a single Lorentz boost and a rotation. This extra rotation experienced by an accelerating particle with non-zero spin is interpreted as due to an effective spin-orbit coupling which changes the energy levels of quantum states and causes extra precession in classical accelerating spinning bodies. A brief derivation of Thomas precession is given in Appendix () following [12]. To derive Thomas precession we need to calculate in SR

$$A_T^{\mathrm{SR}} v(t) \equiv \Lambda(\mathbf{v} + \delta\mathbf{v})\Lambda^{-1}(\mathbf{v})v(t) = (I - \Delta\mathbf{v}\cdot\mathbf{K} - \Delta\boldsymbol{\Omega}\cdot\mathbf{J})v(t), \tag{18}$$

which is discussed in Eq. (41). Here $v(t)$ is the particle's velocity at space-time position $x_0(t)$ at time t and $\Delta\boldsymbol{\Omega}$ is interpreted as Thomas precession. Also $\mathbf{K} \equiv (K_x, K_y, K_z)$ and $\mathbf{J} \equiv (J_x, J_y, J_z)$ are the Lorentz boosts and rotations respectively.

Following the same argument, to determine Thomas precession in VSR the required transformation will be

$$L(v + \delta v)L^{-1}(v)v(t). \tag{19}$$

Now since we have already chosen parameters of $L(u)$ to satisfy Eq. (2), we no longer have any more freedom in the choice of parameters. Hence we require to calculate the following transformation

$$A_T^{\mathrm{VSR}} = L(v+\delta v)L^{-1}(v), \tag{20}$$

where $L(v+\delta v)v_0 = v(t+\delta t)$ and $L(v)v_0 = v(t)$ are the VSR transformation matrices which take the electron from its rest frame to the rest frame of the nucleus at times $t+\delta t$ and t respectively. Using the form of $L(v)$ given in Eq. (3) we can calculate A_T^{VSR} in first order of δv_i which turns out to have the form

$$\begin{aligned} A_T^{\mathrm{VSR}} &= \begin{pmatrix} 1 & \gamma^2\delta v_x & \delta v_y & -\gamma^2 v_x \delta v_x \\ \gamma^2 \delta v_x & 1 & 0 & -\gamma^2 \delta v_x \\ \delta v_y & 0 & 1 & -\delta v_y \\ -\gamma^2 v_x \delta v_x & \gamma^2 \delta v_x & \delta v_y & 1 \end{pmatrix} \\ &= I - \Delta\mathbf{v}_{\mathrm{VSR}} \cdot \mathbf{K} - \Delta\mathbf{\Omega}_{\mathrm{VSR}} \cdot \mathbf{J}, \end{aligned} \tag{21}$$

where $\Delta\mathbf{v}_{\mathrm{VSR}} = -\left(\gamma^2\delta v_x, \delta v_y, -\gamma^2 v_x \delta v_x\right)$ and $\Delta\mathbf{\Omega}_{\mathrm{VSR}} = (-\delta v_y, \gamma^2 \delta v_x, 0)$. Following Eq. (46) the angular velocity of the electron will be

$$\boldsymbol{\omega}_{\mathrm{VSR}} = -\frac{\Delta\mathbf{\Omega}_{\mathrm{VSR}}}{\delta t} = \left(a_y, -\gamma^2 a_x, 0\right). \tag{22}$$

In a circular orbit the acceleration is always radial, hence $a_x = 0$ for instantaneous velocity in x direction. Therefore the precession frequency in VSR turns out to be

$$\boldsymbol{\omega}_{\mathrm{VSR}} = (a_y, 0, 0), \tag{23}$$

It can be seen from the above equation that in this case there are rotations around x axis but no rotation around z axis. The spin-orbit coupling term due to this VSR precession is

$$\begin{aligned} U_{\mathrm{VSR}} &= \boldsymbol{\omega}_{\mathrm{VSR}} \cdot \mathbf{s} = s_x a_y \\ &= s_x y \frac{1}{mr}\frac{dV}{dr}. \end{aligned} \tag{24}$$

Hence the total spin interaction energy in case of VSR would be

$$H_{SO}^{\mathrm{VSR}} = \frac{g}{2m^2}\mathbf{s}\cdot\mathbf{L}\frac{1}{r}\frac{dV}{dr} + s_x y \frac{1}{mr}\frac{dV}{dr}. \tag{25}$$

If the electron has a spin state where $\langle s_x \rangle = 0$ then the contribution to the spin-orbital energy of the electron will come only from Eq. (49) and will turn out to be too large by a factor of 2 compared with the experimental results.

THOMAS-BMT EQUATION OF SPIN PRECESSION IN SR AND VSR

In a macroscopic setting, such as particle accelerators, Thomas precession plays a significant role in the precession of particles circulating in an external magnetic field. The total precession frequency is from a combination of Larmor frequency due to the particles magnetic moment and the Thomas precession due to the acceleration involved in the circular motion. The BMT equation [11] which governs the spin precession in an external field is tested in accelerators where the precession rate of particles is measured to determine their anomalous magnetic moments. In this section we show that the Thomas precession in VSR theories derived earlier also modifies the BMT equation and leads to a precession frequency of particles which is not observed.

The equation of motion in an external magnetic field $\mathbf{B}$, in the lab-frame can be given by Eq. (47), where now we have

$$\begin{aligned} \left.\frac{d\mathbf{s}}{dt}\right|_{\text{lab}-\text{frame}} &= \frac{1}{\gamma}\left.\frac{d\mathbf{s}}{d\tau}\right|_{\text{e}-\text{frame}} + \boldsymbol{\omega}_T \times \mathbf{s} \\ &= (\boldsymbol{\omega}_L + \boldsymbol{\omega}_T) \times \mathbf{s}, \end{aligned} \tag{26}$$

where τ is the proper time in particle's rest frame and $\boldsymbol{\omega}_L \equiv -\frac{ge}{2m\gamma}\mathbf{B}'$ and $\mathbf{B}' \equiv \gamma\mathbf{B}_\perp + \mathbf{B}_\parallel$ is the effective magnetic field realized by the particle in its rest frame and the parallel and perpendicular components are with respect to the instantaneous velocity of the particle. For simplicity we take the applied external magnetic field in the z direction and the instantaneous velocity in the x direction as considered while discussing Thomas precession and therefore we get

$$\boldsymbol{\omega}_L = -\frac{ge}{2m}B_z\widehat{\mathbf{k}}. \tag{27}$$

The frequency arising due to Thomas precession ($\boldsymbol{\omega}_T$) can be obtained from Eq. (46). A particle moving in circular orbit in $x-y$ plane under the influence of an external magnetic field will have an acceleration

$$\mathbf{a} = \frac{e}{m\gamma}(\mathbf{v}\times\mathbf{B}) = -\frac{e}{m\gamma}v_x B_z\widehat{\mathbf{j}}, \tag{28}$$

which yields

$$\boldsymbol{\omega}_T = (\gamma-1)\frac{e}{m\gamma}B_z\widehat{\mathbf{k}}. \tag{29}$$

Therefore the total precession of the charged particle is

$$\begin{aligned}\boldsymbol{\omega}_{\text{total}} &= \boldsymbol{\omega}_L + \boldsymbol{\omega}_T \\ &= -\frac{e}{m}\left(\frac{g-2}{2}+\frac{1}{\gamma}\right) B_z \widehat{\mathbf{k}},\end{aligned} \tag{30}$$

in accordance with Thomas-BMT equation [11] (when the applied magnetic field is only in the perpendicular direction). The polarization of relativistic particles ($\gamma \to \infty$) circulating in a transverse magnetic field precesses with a frequency

$$|\boldsymbol{\omega}_{\text{total}}| = \frac{eB_z}{m}\left|\frac{g}{2}-1\right|, \tag{31}$$

which is used for measuring the anomalous magnetic moment of particles.

In the case of VSR the total precession turns out to be

$$\boldsymbol{\omega}^{\text{VSR}}_{\text{total}} = \boldsymbol{\omega}_L + \boldsymbol{\omega}_{\text{VSR}}, \tag{32}$$

where $\boldsymbol{\omega}_{\text{VSR}}$ has the form given in Eq. (23) and using Eq. (28) one gets

$$\boldsymbol{\omega}_{\text{VSR}} = -\frac{e}{m}\frac{\sqrt{\gamma^2-1}}{\gamma^2} B_z \widehat{\mathbf{i}}. \tag{33}$$

Therefore in VSR the total precession frequency of the particle will be

$$\boldsymbol{\omega}^{\text{VSR}}_{\text{total}} = -\frac{e}{m}\frac{\sqrt{\gamma^2-1}}{\gamma^2} B_z \widehat{\mathbf{i}} - \frac{ge}{2m} B_z \widehat{\mathbf{k}}, \tag{34}$$

So according to VSR theories particles circulating in a transverse magnetic field will precess with a frequency which for high energy particles is

$$\left|\boldsymbol{\omega}^{\text{VSR}}_{\text{total}}\right| = \left(\frac{eB_z}{m}\right)\frac{g}{2}, \tag{35}$$

which is too large by a factor of 10^3 compared to observations of the precession rates of electrons and muons which is accurately described by Eq. (31).

CONCLUSION

We check that VSR theories reproduce the result of SR in case of relativistic velocity addition. We show however that VSR theories fail to reproduce one classic result of SR namely Thomas precession which results in the spin orbit coupling interaction predicted by

VSR theories to be too large by a factor of 2 compared to observations of the fine structure of atomic spectra. It is also interesting to note that as there is no Lorentz violating parameter in VSR theories, it is not possible to tune the parameter such that by doing so one can obtain Thomas precession in these theories. It is the structure of the proper subgroups of Lorentz group which leads to yield incorrect Thomas precession in VSR theories.

Lorentz transformations have the property that two successive Lorentz boosts is equivalent to a boost and a rotation, from the Lorentz algebra $[K_i, K_j] = -\epsilon_{ijk} J_k$. The algebra of $HOM(2)$ or $SIM(2)$ is different so in the dynamics two successive VSR boosts cannot be expressed as a combination of a VSR boost and VSR rotation. Since accelerating observers have to be expressed in terms of two separate boosts at t and $t+\delta t$, the results for accelerated observers differ between Lorentz transformation and VSR transformations.

We also show that the equivalent of the BMT equation derived from VSR theories results predicts the precession frequency of highly relativistic particles in an external magnetic field to be too large by a factor of 10^3 compared to observations.

We conclude that although VSR can be used to derive many of the classic results of Special Relativity it fails to give the correct result for Thomas precession and is therefore ruled out as a fundamental symmetry principle on which field theory of particles can be constructed.

Thomas precession and spin-orbital coupling in SR

In SR an instantaneous acceleration can be mimicked by a Lorentz transformation combined with a rotation. Consider an electron moving in an orbit in $x - y$ plane around a nucleus. Let the velocity of the electron in the rest frame of the nucleus be $\mathbf{v} = (v_x, 0, 0)$ at some time t and at a later time $t + \delta t$ the velocity be $\mathbf{v} + \delta\mathbf{v} = (v_x + \delta v_x, \delta v_y, 0)$. In SR there is a Lorentz transformation which connects the instantaneous electron velocity at time t to its velocity $v_0 = (1, 0, 0, 0)$ in its own rest frame,

$$\Lambda(\mathbf{v}) v_0 = v(t) \tag{36}$$

and similarly another Lorentz transformation connects the electron velocity at time $t + \delta t$ with its velocity in its rest frame v_0,

$$\Lambda(\mathbf{v} + \delta\mathbf{v}) v_0 = v(t + \delta t). \tag{37}$$

The two velocities at different times can be connected with the Lorentz transformation matrix $A_T^{\rm SR}$,

$$v(t+\delta t) = A_T^{\rm SR} v(t), \tag{38}$$

which using Eq. (36) and Eq. (37) gives us

$$A_T^{\rm SR} = \Lambda(\mathbf{v}+\delta\mathbf{v})\Lambda^{-1}(\mathbf{v}). \tag{39}$$

This matrix in the first order in $\delta\mathbf{v}$ gives us [12],

$$A_T^{\rm SR} = \begin{pmatrix} 1 & \gamma^2\delta v_x & \gamma\delta v_y & 0 \\ \gamma^2\delta v_x & 1 & \frac{\gamma-1}{v_x}\delta v_y & 0 \\ \gamma\delta v_y & -\frac{\gamma-1}{v_x}\delta v_y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{40}$$

which can be written in terms of J_i and K_i as

$$A_T^{\rm SR} = I - \Delta\mathbf{v}\cdot\mathbf{K} - \Delta\boldsymbol{\Omega}\cdot\mathbf{J}, \tag{41}$$

where $\Delta\mathbf{v} = -\left(\gamma^2\delta v_x, \gamma\delta v_y, 0\right)$ and $\Delta\boldsymbol{\Omega} = \left(0, 0, \frac{\gamma-1}{v_x}\delta v_y\right)$. To first order in δv_i the above equation can be written as

$$A_T^{\rm SR} = A_{\rm boost}\left(\Delta\mathbf{v}\right) R\left(\Delta\boldsymbol{\Omega}\right) = R\left(\Delta\boldsymbol{\Omega}\right) A_{\rm boost}\left(\Delta\mathbf{v}\right), \tag{42}$$

where

$$A_{\rm boost}\left(\Delta\mathbf{v}\right) = I - \Delta\mathbf{v}\cdot\mathbf{K}, \tag{43}$$

$$R\left(\Delta\boldsymbol{\Omega}\right) = I - \Delta\boldsymbol{\Omega}\cdot\mathbf{J}, \tag{44}$$

and the rotation angle

$$\Delta\boldsymbol{\Omega} = \left(0, 0, \frac{\gamma-1}{v_x}\delta v_y\right). \tag{45}$$

So the electron rotates with respect to the frame of the nucleus with an angular velocity

$$\boldsymbol{\omega}_T = -\frac{\Delta\boldsymbol{\Omega}}{\delta t} = \left(0, 0, -\frac{\gamma^2}{\gamma+1}v_x a_y\right) \simeq \left(0, 0, -\frac{1}{2}v_x a_y\right), \tag{46}$$

where the last equality is obtained assuming non-relativistic limits. The spin of the electron precesses in the rest frame of the nucleus as

$$\left.\frac{d\mathbf{s}}{dt}\right|_{\rm nucleus-frame} = \left.\frac{d\mathbf{s}}{dt}\right|_{\rm e-frame} + \boldsymbol{\omega}_T\times\mathbf{s}. \tag{47}$$

This extra precession known as the Thomas precession corresponds to an interaction

$$U_T = \boldsymbol{\omega}_T \cdot \mathbf{s}. \tag{48}$$

The electron's magnetic moment has the interaction energy

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}', \tag{49}$$

where $\boldsymbol{\mu} = \frac{ge}{2m}\mathbf{s}$ is the magnetic moment of the electron and $\mathbf{B}' = -\mathbf{v} \times \mathbf{E}$ is the effective magnetic field in the rest frame of the electron and $\mathbf{E}$ is the electric field of the nucleus given as

$$\mathbf{E} = -\frac{\mathbf{r}}{er}\frac{dV}{dr}. \tag{50}$$

The total spin-orbital interaction energy of the electron is therefore

$$H_{SO} = -\boldsymbol{\mu} \cdot \mathbf{B}' + \boldsymbol{\omega}_T \cdot \mathbf{s}. \tag{51}$$

Using the fact that $a_y = eE_y/m$, the spin-orbital energy turns out to be

$$H_{SO} = \frac{g-1}{2m^2}\mathbf{s} \cdot \mathbf{L}\frac{1}{r}\frac{dV}{dr}. \tag{52}$$

The measurement of the Zeeman-splitting of spectral lines in a magnetic field shows that the gyromagnetic ratio of the electron is $g \simeq 2$. If the Thomas precession was absent the spin-orbital coupling term would have a factor of g in the first bracket instead of $(g-1)$ which would have resulted in a factor of 2 discrepancy with observations.

---

[1] A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. **97**, 021601 (2006) [arXiv:hep-ph/0601236].

[2] A. G. Cohen and S. L. Glashow, arXiv:hep-ph/0605036.

[3] M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. **101**, 261601 (2008) [arXiv:0806.3699 [hep-th]].

[4] G. W. Gibbons, J. Gomis and C. N. Pope, Phys. Rev. D **76**, 081701 (2007) [arXiv:0707.2174 [hep-th]].

[5] E. Alvarez and R. Vidal, Phys. Rev. D **77**, 127702 (2008) [arXiv:0803.1949 [hep-th]].

[6] W. Muck, Phys. Lett. B **670**, 95 (2008) [arXiv:0806.0737 [hep-th]].

[7] A. G. Cohen and D. Z. Freedman, JHEP **0707**, 039 (2007) [arXiv:hep-th/0605172].

[8] U. Lindstrom and M. Rocek, arXiv:hep-th/0606093.

[9] A. Katz and Y. Shadmi, Phys. Rev. D **74**, 115021 (2006) [arXiv:hep-ph/0605210].

[10] L.H. Thomas, Phil. Mag. **3**, 1 (1927).

[11] V. Bargmann, L. Michel and V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).

[12] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Sec. 11.5.